Fourier series

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In this lecture, we focus on the vector space $L^2_{T_0}(\mathbb{R},\mathbb{C})$ of periodic signals with period T_0 of finite average power, equipped with the Hermitian product $\langle x, y\rangle_{T_0} = \dfrac{1}{T_0}\displaystyle\int_0^{T_0} x(t)y^*(t)dt$. With the linear review of the previous lecture, we can now dive into the structure of this vector space.

Proposition 0.1
For any $n \in \mathbb{N}$, the set of complex exponentials $e_{-n\omega_0}, \dots, e_{n\omega_0}$ is an orthonormal set of $L^2_{T_0}(\mathbb{R},\mathbb{C})$, thus an orthonormal basis of the subspace $\text{Span}(e_{-n\omega_0}, \dots, e_{n\omega_0})$ of $L^2_{T_0}(\mathbb{R},\mathbb{C})$.

PROOF : Let two integers j and k of $[\![-n, n]\!]$ such that $j \neq k$. Then

$$\langle e_{j\omega_0}, e_{k\omega_0}\rangle_{T_0} = \frac{1}{T_0}\int_0^{T_0} e_{j\omega_0}(t)\overline{e_{k\omega_0}(t)}dt = \frac{1}{T_0}\int_0^{T_0} e^{ij\omega_0 t}e^{-ik\omega_0 t}dt = \frac{1}{T_0}\left[\frac{e^{i(j-k)\omega_0 t}}{i(j-k)\omega_0}\right]_0^{T_0} = 0$$

For any $j \in [\![-n, n]\!]$,

$$\langle e_{j\omega_0}, e_{j\omega_0}\rangle_{T_0} = \frac{1}{T_0}\int_0^{T_0} e_{j\omega_0}(t)\overline{e_{j\omega_0}(t)}dt = \frac{1}{T_0}\int_0^{T_0} dt = 1$$

thus $\|e_{j\omega_0}\|_{T_0} = \sqrt{\langle e_{j\omega_0}, e_{j\omega_0}\rangle_{T_0}} = 1$, yielding the result. ∎

Definition 0.1 (Fourier coefficients, Fourier series, partial sum)
Let x be a signal of $L^2_{T_0}(\mathbb{R},\mathbb{C})$. The **Fourier coefficients** of x are the complex numbers defined by:

$$\forall n \in \mathbb{Z} \qquad c_n(x) = \langle x, e_{n\omega_0}\rangle_{T_0} = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-in\omega_0 t}dt$$

The **Fourier series** of x is the function

$$t \mapsto \sum_{n=-\infty}^{+\infty} c_n(x)e^{in\omega_0 t} = \sum_{n=-\infty}^{+\infty} \langle x, e_{n\omega_0}\rangle_{T_0} e_{n\omega_0}(t)$$

The **partial sum** of index N of the Fourier series of x is its projection onto the subspace $\text{Span}(e_{-N\omega_0}, \dots, e_{N\omega_0})$, i.e.

$$S_N(x) = \sum_{n=-N}^{N} \langle x, e_{n\omega_0}\rangle_{T_0} e_{n\omega_0} = \sum_{n=-N}^{N} c_n(x)e_{n\omega_0}$$

Remark: In particular,

$$c_0(x) = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

is the average value of x over the interval $[0, T_0]$. This coefficient is called the **DC component** of x, as opposition with the other coefficients called the AC (alternating current) component. A zero-mean signal x is such that $c_0(x) = 0$.

Definition 0.2 (Trigonometric polynomial)
A **trigonometric polynomial** is any function of the form:

$$P : \mathbb{R} \to \mathbb{C} \qquad t \mapsto \sum_{n=-N}^{N} c_n e^{in\omega_0 t} = \sum_{n=-N}^{N} c_n \left(e^{i\omega_0 t}\right)^n$$

where $N \in \mathbb{N}$ and $(c_n)_{n \in [\![-N,N]\!]}$ is a sequence of complex numbers, i.e. a polynomial in the complex exponential $e^{i\omega_0 t}$.

Remark: According to this definition, the partial sum of a Fourier series is a trigonometric polynomial. Weierstrass' theorem states that any continuous function defined on a compact interval of $\mathbb{R}$ is the uniform limit of a sequence of polynomials. This theorem can be adapted to the space $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ by stating that any continuous periodic function with period T_0 is the uniform limit of a sequence of trigonometric polynomials, which implies in this context that any continuous periodic signal is equal to its Fourier series.

Definition 0.3 (Convergence in quadratic mean)
A sequence $(x_n)_{n\in\mathbb{N}}$ of signals in $L^2_{T_0}(\mathbb{R}, \mathbb{C})$ **converges in quadratic mean** to $x \in L^2_{T_0}(\mathbb{R}, \mathbb{C})$ if

$$\lim_{n\to+\infty} \|x_n - x\|^2_{T_0} = \lim_{n\to+\infty} \frac{1}{T_0} \int_0^{T_0} |x_n(t) - x(t)|^2 dt = 0$$

Definition 0.4 (Pre-Hilbert space, Hilbert space, Hilbert basis)

- A **pre-Hilbert space** is a vector space equipped with a scalar product or a Hermitian product.
- A **Hilbert space** is a complete pre-Hilbert space, meaning that any Cauchy sequence has a limit in this space.
- A **Hilbert basis** of a Hilbert space V is a sequence $(v_n)_{n\in\mathbb{N}}$ of vectors of V satisfying the following properties:
  - for any $n \in \mathbb{N}$, $\|v_n\| = 1$, and for any $(n, m) \in \mathbb{N}^2$ such that $n \neq m$, $\langle v_n, v_m \rangle = 0$;
  - the set $\operatorname{Span}((v_n)_{n\in\mathbb{N}})$ is dense in V, i.e. any vector $v \in V$ is the limit of the sequence of vectors in $\operatorname{Span}((v_n)_{n\in\mathbb{N}})$.

Example 0.1
Spaces $L^2(\mathbb{R}, \mathbb{K})$ and $L^2_{T_0}(\mathbb{R}, \mathbb{K})$ are infinite-dimensional Hilbert spaces. Their completeness is ensured by the Riesz-Fischer theorem.

Remarks:

- The equivalence of norms in finite dimension ensures that any finite-dimensional vector space is complete, so that any finite-dimensional pre-Hilbert space is a Hilbert space. In infinite dimension, some spaces may be pre-Hilbert without being Hilbert. The Hilbert space structure enables an easy handling of scalar products in infinite dimension.
- Note the difference between a Hilbert basis and an algebraic orthonormal basis defined in the previous lecture.

Theorem 0.2
The set of complex exponentials $(e_{n\omega_0})_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2_{T_0}(\mathbb{R}, \mathbb{C})$.

PROOF : The proof is out of the scope of this document. ■

Theorem 0.3 (Parseval's theorem)

Let a periodic signal $x \in L^2_{T_0}(\mathbb{R}, \mathbb{C})$ whose series of Fourier coefficients $(c_n(x))_{n\in\mathbb{Z}}$ is absolutely convergent, i.e. $\sum_{n=-\infty}^{+\infty} |c_n(x)| < +\infty$. Then the corresponding Fourier series converges in quadratic mean to x, i.e.

$$\lim_{N\to+\infty} \frac{1}{T_0} \int_0^T |S_N(x)(t) - x(t)|^2 dt = 0$$

Moreover, Parseval's identity states:

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |c_n(x)|^2$$

PROOF : Let $x \in L^2_{T_0}(\mathbb{R}, \mathbb{C})$. Since $(e_{n\omega_0})_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2_{T_0}(\mathbb{R}, \mathbb{C})$, we can write

$$x = \sum_{n=-\infty}^{+\infty} \langle x, e_{n\omega_0} \rangle_{T_0} e_{n\omega_0} = \sum_{n=-\infty}^{+\infty} c_n(x) e_{n\omega_0} = S_N(x) + \sum_{|n|>N} c_n(x) e_{n\omega_0}$$

By triangular inequality, we get:

$$\|S_N(x) - x\|_{T_0} \leq \sum_{|n|>N} |c_n(x)|$$

From the absolute convergence of Fourier coefficients, we deduce that $\lim_{N\to+\infty} \|S_N(x) - x\|_{T_0} = 0$.
Parseval's identity is simply the Pythagorean theorem for Hilbert spaces. Indeed,

$$\frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt = \|x\|^2_{T_0} = \langle x, x \rangle_{T_0} = \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} c_n(x) \overline{c_m(x)} \langle e_{n\omega_0}, e_{m\omega_0} \rangle = \sum_{n=-\infty}^{+\infty} |c_n(x)|^2$$ ■

Proposition 0.4

Fourier series satisfy the following properties:

(i) linearity: for two periodic signals x and y with period T_0, if $z = \alpha x + \beta y$, then for any $n \in \mathbb{Z}$,

$$c_n(z) = \alpha c_n(x) + \beta c_n(y)$$

(ii) differentiation: for any $n \in \mathbb{Z}$, $c_n(x') = in\omega_0 c_n(x)$;

(iii) pure delay: for any $a \in \mathbb{R}$, for any $n \in \mathbb{Z}$, $c_n(\tau_a(x)) = e^{-in\omega_0 a} c_n(x)$

(iv) symmetry: if $\tilde{x} : t \mapsto x(-t)$, then for any $n \in \mathbb{Z}$, $c_n(\tilde{x}) = (c_n(x^*))^*$.

(iv) circular convolution: for two periodic signals x and y with period T_0, for any $n \in \mathbb{Z}$, $c_n(x \otimes y) = c_n(x) c_n(y)$.

PROOF : (i) By linearity of the Hermitian product in the first component,

$$c_n(z) = c_n(\alpha x + \beta y) = \langle \alpha x + \beta y, e_{n\omega_0} \rangle = \alpha \langle x, e_{n\omega_0} \rangle + \beta \langle y, e_{n\omega_0} \rangle = \alpha c_n(x) + \beta c_n(y)$$

(ii) An integration by parts yields:

$$c_n(x') = \frac{1}{T_0}\int_0^{T_0} x'(t)e^{-in\omega_0 t}dt = \frac{1}{T_0}\Big[x(t)e^{-in\omega_0 t}\Big]_0^{T_0} + \frac{in\omega_0}{T_0}\int_0^{T_0} x(t)e^{-in\omega_0 t}dt = in\omega_0 c_n(x)$$

(iii) By the change of variable $t \mapsto t - a$, we get:

$$c_n(\tau_a(x)) = \frac{1}{T_0}\int_0^{T_0} x(t-a)e^{-in\omega_0 t}dt = \frac{1}{T_0}\int_{-a}^{-a+T_0} x(t)e^{-in\omega_0 (t+a)}dt = e^{-in\omega_0 a}c_n(x)$$

(iv) By the change of variable $t \mapsto -t$, we get:

$$c_n(\tilde{x}) = \frac{1}{T_0}\int_0^{T_0} x(-t)e^{-in\omega_0 t}dt = \frac{1}{T_0}\int_{-T_0}^{0} x(t)e^{in\omega_0 t}dt = \frac{1}{T_0}\left(\int_0^{T_0} x^*(t)e^{-in\omega_0 t}dt\right)^* = (c_n(x^*))^*$$

(v) As seen in the lecture about periodic signals, circular convolution

$$\forall t \in \mathbb{R} \qquad (x \otimes y)(t) = \frac{1}{T_0}\int_0^{T_0} x(u)y(t-u)du$$

is also a periodic signal with period T_0. Its Fourier coefficients are

$$c_n(x \otimes y) = \frac{1}{T_0}\int_0^{T_0} (x \otimes y)(t)e^{-in\omega_0 t}dt = \frac{1}{T_0^2}\int_0^{T_0}\int_0^{T_0} x(u)e^{-in\omega_0 u}y(t-u)e^{-in\omega_0 (t-u)}dudt$$

Bu the change of variable $(t, u) \mapsto (t + u, u)$ and by Fubini's theorem,

$$c_n(x \otimes y) = \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-in\omega_0 t}dt\right)\left(\frac{1}{T_0}\int_0^{T_0} y(u)e^{-in\omega_0 u}du\right) = c_n(x)c_n(y)$$ ■

Remark: Property (ii) shows that for any differentiable signal x, $c_0(x') = 0$ therefore the derivative of a periodic signal is a zero-mean periodic signal. Conversely, it implies that we can only integrate zero-mean periodic signals, otherwise the antiderivative would have a term $t \mapsto c_0(x).t$ which cannot be written as a Fourier series since it is not periodic signal.

We turn to the particular case of real-valued periodic signals and express their Fourier series as sums of sines and cosines.

Lemma 0.5
If $x \in L^2_{T_0}(\mathbb{R}, \mathbb{R})$ is a real-valued signal, then for any $n \in \mathbb{Z}$, $c_{-n}(x) = c_n(x)^*$. In particular, $c_0(x) \in \mathbb{R}$.

Proof : If $x \in L^2_{T_0}(\mathbb{R}, \mathbb{R})$, then for any $t \in \mathbb{R}$, $x^*(t) = x(t)$, thus for any $n \in \mathbb{Z}$,

$$c_{-n}(x) = \frac{1}{T_0}\int_0^{T_0} x(t)e^{in\omega_0 t}dt = \left(\frac{1}{T_0}\int_0^{T_0} x(t)e^{-in\omega_0 t}dt\right)^* = c_n(x)^*$$

In particular, for $n = 0$, $c_0(x) = c_0(x)^*$, thus $c_0(x) \in \mathbb{R}$. ■

Proposition 0.6
Any real-valued signal $x \in L^2_{T_0}(\mathbb{R}, \mathbb{R})$ can be written:

$$\forall t \in \mathbb{R} \qquad x(t) = \sum_{n=0}^{+\infty} a_n(x)\cos(n\omega_0 t) + \sum_{n=1}^{+\infty} b_n(x)\sin(n\omega_0 t)$$

with

$$a_0(x) = \frac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \forall n \in \mathbb{N}^* \quad a_n(x) = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega_0 t)dt \quad b_n(x) = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega_0 t)dt$$

PROOF : Using identity $e^{in\omega_0 t} = \cos(n\omega_0 t) + i\sin(n\omega_0 t)$, we can write:

$$\begin{aligned}
x(t) = \sum_{n=-\infty}^{+\infty} c_n(x)e^{in\omega_0 t} &= c_0(x) + \sum_{n=1}^{+\infty} c_n(x)e^{in\omega_0 t} + \sum_{n=1}^{+\infty} c_{-n}(x)e^{-in\omega_0 t} \\
&= c_0(x) + \sum_{n=1}^{+\infty} c_n(x)\left(\cos(n\omega_0 t) + i\sin(n\omega_0 t)\right) + \sum_{n=1}^{+\infty} c_n(x)^*\left(\cos(n\omega_0 t) - i\sin(n\omega_0 t)\right) \\
&= c_0(x) + \sum_{n=1}^{+\infty}\left(c_n(x) + c_n(x)^*\right)\cos(n\omega_0 t) + \sum_{n=1}^{+\infty} i\left(c_n(x) - c_n(x)^*\right)\sin(n\omega_0 t)
\end{aligned}$$

We obtain the expected form by setting:

$$a_0(x) = c_0(x) = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_n(x) = c_n(x) + c_n(x)^* = \frac{1}{T_0}\int_0^{T_0} x(t)\left(e^{-in\omega_0 t} + e^{in\omega_0 t}\right)dt = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(n\omega_0 t)dt$$

$$b_n(x) = i\left(c_n(x) - c_n(x)^*\right) = \frac{i}{T_0}\int_0^{T_0} x(t)\left(e^{-in\omega_0 t} - e^{in\omega_0 t}\right)dt = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(n\omega_0 t)dt$$ ■

Remark: The formulation of Fourier series as sums of sines and cosines can be useful for real-valued even or odd signals.

Proposition 0.7 (Parseval's identity)
For any real-valued signal $x \in L^2_{T_0}(\mathbb{R}, \mathbb{R})$, Parseval's identity becomes:

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = a_0(x)^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(a_n(x)^2 + b_n(x)^2)$$

PROOF : Using the expressions $a_n(x)$ and $b_n(x)$ as functions of $c_n(x)$, we have:

$$\forall n \in \mathbb{N}^* \qquad c_n(x) = \frac{a_n(x) - ib_n(x)}{2} \qquad c_n(x)^* = \frac{a_n(x) + ib_n(x)}{2}$$

so that

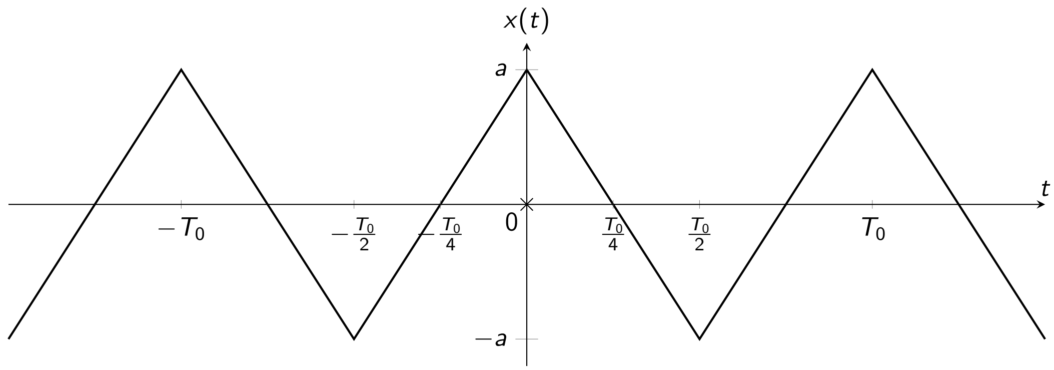
$$|c_n(x)|^2 = c_n(x)c_n(x)^* = \frac{a_n(x)^2 + b_n(x)^2}{4}$$

For any $n < 0$, $c_n(x) = c_{-n}(x)^*$, thus $|c_n(x)|^2 = |c_{-n}(x)|^2$. With the general Parseval's formula, we get:

$$\frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |c_n(x)|^2 = |c_0|^2 + 2\sum_{n=1}^{+\infty} |c_n(x)|^2 = |c_0|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}(a_n(x)^2 + b_n(x)^2)$$ ■

Example 0.2

We compute the Fourier series of the following **triangle wave** signal:



Signal x is periodic with period T_0, it is even, and for any $t \in \left[0, \frac{T_0}{2}\right]$, $x(t) = a\left(1 - \frac{4}{T_0}t\right)$. Since it is even, its Fourier series can be written as:

$$x(t) = a_0(x) + \sum_{n=1}^{+\infty} a_n(x)\cos(n\omega_0 t)$$

On one hand,

$$a_0(x) = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt = \frac{2a}{T_0}\int_0^{\frac{T_0}{2}} \left(1 - \frac{4}{T_0}t\right) dt = -\frac{a}{4}\left[\left(1 - \frac{4}{T_0}t\right)^2\right]_0^{\frac{T_0}{2}} = 0$$

meaning that x is a zero-mean signal, this property being visible on its graphical representation. On the other hand, for any $n \in \mathbb{N}^*$,

$$a_n(x) = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)dt = \frac{4a}{T_0}\int_0^{\frac{T_0}{2}} \left(1 - \frac{4}{T_0}t\right)\cos(n\omega_0 t)dt$$

We have

$$\int_0^{\frac{T_0}{2}} \cos(n\omega_0 t)dt = 0 \qquad \text{and} \qquad \int_0^{\frac{T_0}{2}} t\cos(n\omega_0 t)dt = \frac{(-1)^n - 1}{n^2\omega_0^2}$$

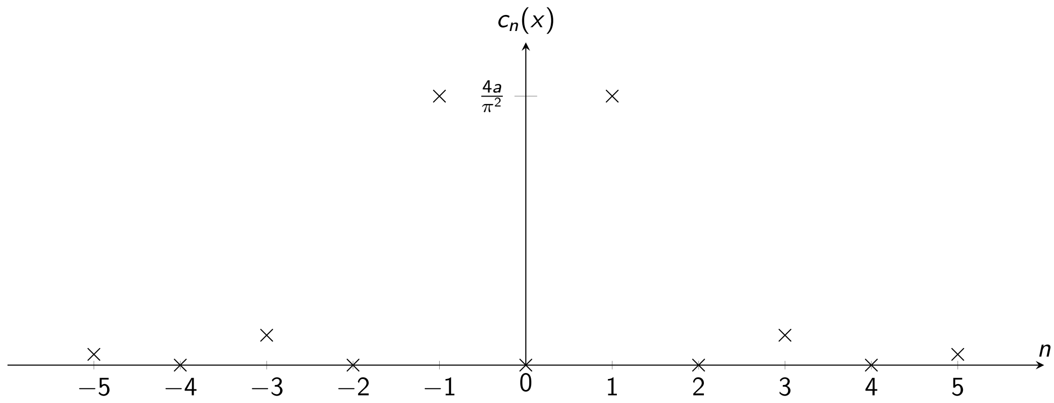
yielding

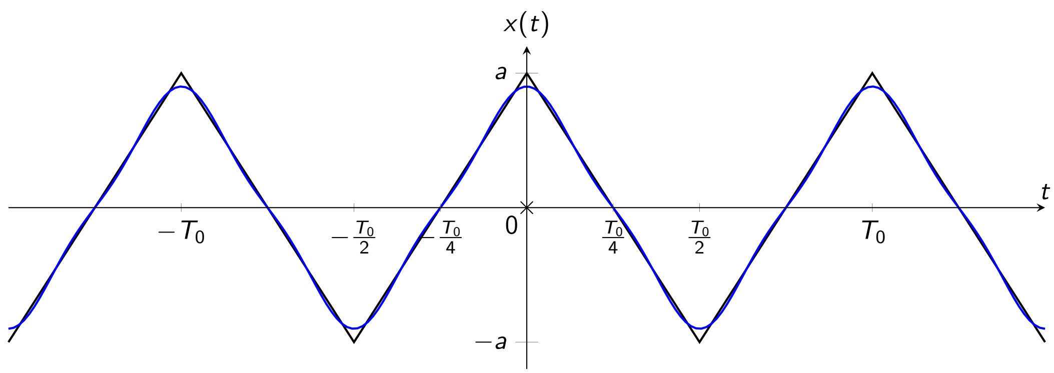
$$a_n(x) = \frac{16a(1-(-1)^n)}{n^2\omega^2 T_0^2} = \frac{4a(1-(-1)^n)}{\pi^2 n^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8a}{\pi^2 n^2} & \text{if } n \text{ is odd} \end{cases}$$

We deduce

$$x(t) = \frac{8a}{\pi^2}\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}\cos\left((2n+1)\omega_0 t\right) = \frac{2a}{\pi^2}\sum_{n=-\infty}^{+\infty} \frac{1-(-1)^n}{n^2} e^{in\omega_0 t}$$

Then we can display the Fourier series by drawing the Fourier coefficients:



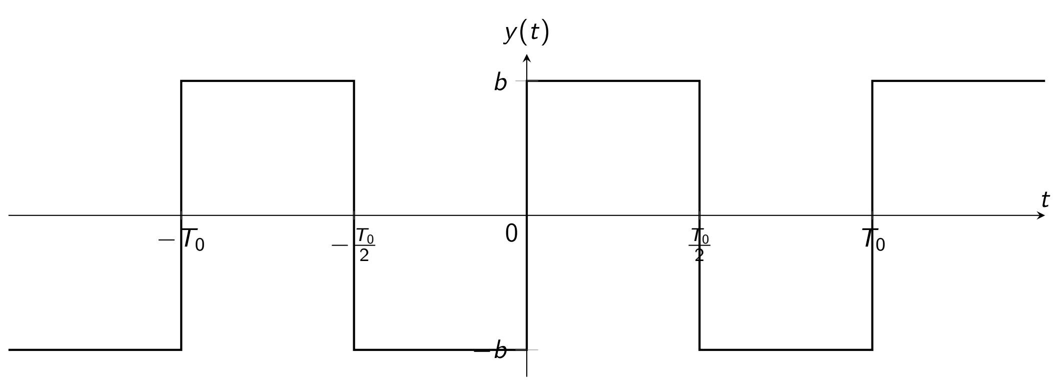


The blue curve represents the partial sum

$$\frac{8a}{\pi^2}\sum_{n=0}^{N}\frac{1}{(2n+1)^2}\cos\left((2n+1)\omega_0 t\right)$$

of $x(t)$ for $N = 1$.

Example 0.3
We compute the Fourier series of the following **square wave** signal:



Signal y is periodic with period T_0, it is odd, for any $t \in \left[0, \frac{T_0}{2}\right[$, $y(t) = b$, and for any $t \in \left[-\frac{T_0}{2}, 0\right[$, $y(t) = -b$.
Since it is odd, its Fourier series can be written as:

$$y(t) = \sum_{n=1}^{+\infty} b_n(y)\sin(n\omega_0 t)$$

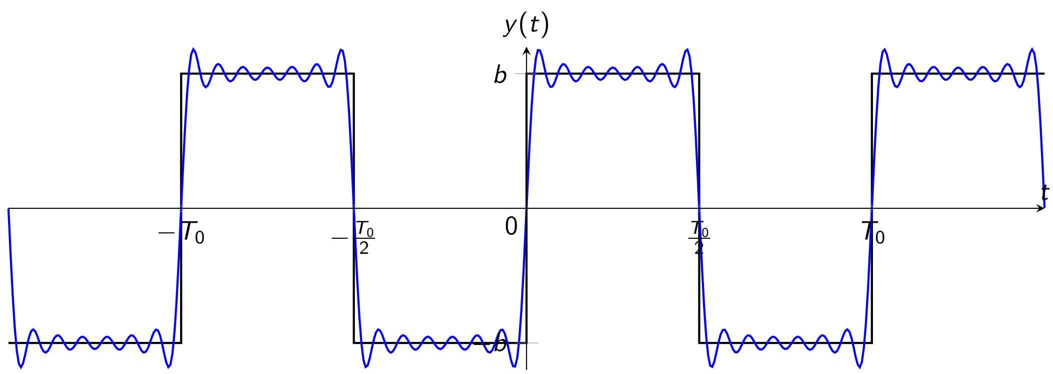
For any $n \in \mathbb{N}^*$,

$$b_n(y) = \frac{2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\sin(n\omega_0 t)dt = \frac{4b}{T_0}\int_0^{\frac{T_0}{2}} \sin(n\omega_0 t)dt = \frac{4b}{T_0}\left[-\frac{\cos(n\omega_0 t)}{n\omega_0}\right]_0^{\frac{T_0}{2}} = \frac{4b}{T_0}\frac{1-(-1)^n}{n\omega_0} = \frac{2b}{\pi}\frac{1-(-1)^n}{n}$$

yielding

$$y(t) = \frac{2b}{\pi}\sum_{n=1}^{+\infty}\frac{1-(-1)^n}{n}\sin(n\omega_0 t) = \frac{4b}{\pi}\sum_{n=0}^{+\infty}\frac{1}{2n+1}\sin((2n+1)\omega_0 t)$$

Consider again the triangle wave signal x from the previous example. Then for any $t \in \left[0, \frac{T_0}{2}\right[$, $y(t) = -\frac{4a}{T_0}$, and for any $t \in \left[-\frac{T_0}{2}, 0\right[$, $y(t) = \frac{4a}{T_0}$. Setting $b = -\frac{4a}{T_0}$, we have $y(t) = x'(t)$. Thus we can also deduce the Fourier series of y from x and use Fourier series differentiation property.



The blue curve represents the partial sum

$$\frac{4b}{\pi}\sum_{n=0}^{+\infty}\frac{1}{2n+1}\sin((2n+1)\omega_0 t)$$

of $y(t)$ for $N = 6$. We notice overshoots near the discontinuities of y. This phenomenon is called **Gibb's phenomenon**.

Proposition 0.8
Consider an LTI system L with impulse response $h = L(\delta)$. If we input a periodic signal x with period T_0 and Fourier coefficients $(c_n(x))_{n\in\mathbb{Z}}$, then the corresponding output $y = L(x)$ is also periodic with period T_0 and its Fourier coefficients satisfy the following identity:

$$\forall n \in \mathbb{Z} \qquad c_n(y) = H(n\omega_0)c_n(x) \qquad \text{where} \quad H(n\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-in\omega_0 t}dt$$

Proof : We have proved in the lecture about periodic signals that if the input of an LTI system is periodic, the corresponding output is also periodic with the same period. We have seen in the previous lecture that complex exponentials are eigenfunctions of any LTI system. Combining this property with the linearity of L, we can write:

$$y = \sum_{n=-\infty}^{+\infty} c_n(y)e_{n\omega_0} = L(x) = \sum_{n=-\infty}^{+\infty} c_n(x)L(e_{n\omega_0}) = \sum_{n=-\infty}^{+\infty} c_n(x)H(n\omega_0)e_{n\omega_0}$$

Since $(e_{n\omega_0})_{n\in\mathbb{Z}}$ is a Hilbert basis of $L^2_{T_0}(\mathbb{R},\mathbb{C})$, the coordinates of any signal of $L^2_{T_0}(\mathbb{R},\mathbb{C})$ in this basis are unique, which implies that for any $n \in \mathbb{Z}$, $c_n(y) = c_n(x)H(n\omega_0)$. ■

Remarks:

- This property implies what we sometimes call the **cos in** / **cos out rule**: if the cosine $c_{\omega_0,A_x,\varphi_x}$ of fundamental impulse ω_0, amplitude A_x and phase φ_x is the input of an LTI system, then the corresponding output is the cosine $c_{\omega_0,A_y,\varphi_y}$ with the same fundamental impulse ω_0, and whose the amplitude and phase are given by the identities $A_y = A_x|H(\omega_0)|$ and $\varphi_y = \varphi_x + \mathrm{Arg}\,(H(\omega_0))$.
- We have seen the interest in LTI systems in the explicit definition of their output as the convolution of the input and the impulse response. However, we have seen in the lecture about periodic signals that the direct computation of convolution can be cumbersome. This proposition is very powerful because we can circumvent this convolution and we can directly compute the output corresponding to a periodic input by the only knowledge of the Fourier coefficients of this input and the coefficients $H(n\omega_0)$. In the next lecture, we extend this method to any signal in $L^2(\mathbb{R},\mathbb{C})$ by the introduction of the Fourier transform.